Calculus I
Notes on Basic Rules for Derivatives.

Power Functions and The Power Rule:
First remember the derivatives we found in previous notes:

<table>
<thead>
<tr>
<th>A Few Basic Derivatives</th>
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</thead>
<tbody>
<tr>
<td>1. ( \frac{d}{dx}[x] = 1 )</td>
</tr>
<tr>
<td>2. ( \frac{d}{dx}[x^2] = 2x )</td>
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Let’s focus in on #1-5. The 6th does follow the same rule, but it harder to see it with this one. So we’ll talk about it later.

Now we need to look at these in a different form.

Rewriting
Pulling out the main parts:

\[
\frac{d}{dx}[x] = \frac{d}{dx}[x^1] = 1 \cdot x^0 \\
\frac{d}{dx}[x^2] = 2x^1 \\
\frac{d}{dx}[x^3] = 3x^2 \\
\frac{d}{dx}[\frac{1}{x}] = \frac{d}{dx}[x^{-1}] = -\frac{1}{x^2} \\
\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{-0.5} = \frac{1}{2}x^{-0.5} \\
\frac{d}{dx}[1] = \frac{d}{dx}[1] = 0
\]

What is the pattern? (DON’T just read on without looking. If you look, it will be much more memorable to whether you found it or not.)
First, all of them can be written as x raised to a power. Then its derivative is that power times x raised to the power minus 1. This pattern holds for all functions of this form, which is called a basic power function.

Basic Power Functions
\[ f(x) \] is a basic power function, if and only if it can be put into the following form:
\[ f(x) = x^p \]
where
x is the variable
p is any real number

Any transformation of the graph you would have learned to this point of a basic power function could be called a power function.

Example #1: Classify the following as a basic power function, a power function, or neither. State why.

a) \( f(x) = 4^x \)  b) \( g(x) = 3 \cdot (x-5)^7 + 7 \)  c) \( h(x) = \sqrt[3]{x^2} \)  d) \( J(x) = \frac{-9}{(x+6)^{7/2}} + 9 \)  e) \( K(x) = x^{e \pi} \)  f) \( L(x) = \sin(x) \)

\( f(x) = 4^x \) is neither since the variable is in the exponent and not the base. It’s a basic exponential function.

b) \( g(x) = 3 \cdot (x-5)^7 + 7 \) is a power function since it is transformations of \( y = x^7 \) which is a basic power function.

\( h(x) = \sqrt[3]{x^2} = x^{2/3} \). So it is a basic power function and thus also a power function.

d) \( J(x) = \frac{-9}{(x+6)^{7/2}} + 9 = -(9(x+6)^{-7/2} + 9) \). Thus, it’s a power function, because it’s transformations of \( y = x^{-7/2} \).

e) \( K(x) = x^{e \pi} \) is a basic power function since the variable is in the base and the exponent, \( \pi + e \), is a real number. Therefore, it’s also a power function.

f) \( L(x) = \sin(x) \) is neither it’s a trigonometric function and can’t be put into the right form.
So definitely not all functions are power functions, and the pattern we noticed earlier won’t always be used to find the derivative, though power functions do comprise of a variety of functions including monomials, root, and some rational functions.

Now let’s get to the pattern we noticed earlier.

<table>
<thead>
<tr>
<th>The Power Rule</th>
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<tbody>
<tr>
<td>If ( f(x) ) is a basic power function, then it’s derivative is its power times ( x ) raised to the power minus 1,</td>
</tr>
<tr>
<td>or to put it another way</td>
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</tbody>
</table>
| \[
\frac{d}{dx} \left[ x^p \right] = p \cdot x^{p-1} 
\] |
| where \( p \) is any real number. |

Right now we can’t do a complete proof for all real \( p \). So we’ll just do a proof for the natural exponents.

Partial proof for \( n=1,2,3,4,... \):

\[ f(x) = x^p \]

For \( f(x + \Delta x) \) remember from Precalculus Algebra that the coefficient of the \( n^{th} \) term of \((a + b)^n\) is given by \( \binom{n}{n-1} \). Thus, the first 2 and the last coefficients are

\[
\begin{align*}
C_1 &= \binom{p}{1-1} = \binom{p}{0} = \frac{p!}{0!(p-0)!} = \frac{p!}{p!} = 1 \\
C_2 &= \binom{p}{2-1} = \binom{p}{1} = \frac{p!}{1!(p-1)!} = \frac{p}{(p-1)!} = p \\
C_{p+1} &= \binom{p}{(p+1)-1} = \binom{p}{p} = \frac{p!}{p!(p-p)!} = \frac{p!}{p!0!} = 1
\end{align*}
\]

It turns out, as you will see, that we don’t need to know the rest of the coefficients. So we’ll just label them as \( C_3, C_4, C_5,..., C_p \).

This gives us.

\[
f(x + \Delta x) = (x + \Delta x)^p = x^p + p \cdot x^{p-1} \cdot \Delta x + C_3 \cdot x^{p-2} \cdot (\Delta x)^2 + C_4 \cdot x^{p-3} \cdot (\Delta x)^3 + ... + (\Delta x)^p
\]

Therefore,

\[
\frac{d}{dx} \left[ x^p \right] = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^p + p \cdot x^{p-1} \cdot \Delta x + C_3 \cdot x^{p-2} \cdot (\Delta x)^2 + C_4 \cdot x^{p-3} \cdot (\Delta x)^3 + ... + (\Delta x)^p - x^p}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{p \cdot x^{p-1} \cdot \Delta x + C_3 \cdot x^{p-2} \cdot (\Delta x)^2 + C_4 \cdot x^{p-3} \cdot (\Delta x)^3 + ... + (\Delta x)^p}{\Delta x}
\]

\[
= p \cdot x^{p-1} + C_3 \cdot x^{p-2} \cdot 0 + C_4 \cdot x^{p-3} \cdot (0)^2 + ... + (0)^{p-1}
\]

\[= p \cdot x^{p-1} \]

\[\square\]

Now let’s go back to that 6th basic derivative from the beginning of these note, \( \frac{d}{dx}[1] = 0 \). Keeping in mind that \( x^0 = 1 \), this derivative fits the power rule since

\[
\frac{d}{dx}[1] = \frac{d}{dx}[x^0] = 0 \cdot x^{0-1} = 0
\]
Example #2: Differentiate the following functions.

\[ a) \quad f(x) = x^{18} \quad b) \quad y = \sqrt[3]{x^{17}} \quad c) \quad y = \frac{1}{x^7} \quad d) \quad J(x) = x^x \quad e) \quad y = \frac{x^7}{x^5} \]

First note that differentiate means to “take the derivative of”.

\[ a) \quad f'(x) = 18x^{18-1} = 18x^{17} \]

b) Since the function isn’t in the \( x^p \) form we first have to rewrite it.

\[ y = \sqrt[6]{x^{11}} = x^{\frac{11}{6}} \]

\[ y' = \frac{11}{6} x^{\frac{11}{6} - 1} = \frac{11}{6} x^{\frac{5}{6}} = \frac{11}{6} \sqrt[6]{x^5} \]

Note that the function should be put back into radical form since the original was in radical form.

c) Again we have to rewrite before we can differentiate.

\[ y = \frac{1}{x^7} = x^{-7} \]

\[ \frac{dy}{dx} = -7x^{-7-1} = -7x^{-8} = \frac{-7}{x^8} \]

d) \( J'(x) = \pi x^{x-1} \)

e) \[ y = \frac{x^7}{x^5} = x^{7-5} = x^2 \]

\[ y' = 5x^4 \]

\[ \square \]

Example #3: Find the equation of the tangent line to \( y = \sqrt{x} \) at \( x = 64 \).

Since we need the equation of a line, we need the slope and a point on the line.

\[ \text{slope} = \left. \frac{dy}{dx} \right|_{x=64} = \left. \frac{d}{dx} \left( \sqrt{x} \right) \right|_{x=64} = \frac{d}{dx} \left( x^{\frac{1}{2}} \right) \bigg|_{x=64} = \frac{1}{2} x^{\frac{1}{2}-1} \bigg|_{x=64} = \frac{1}{2} \left( \sqrt{64} \right) \bigg|_{x=64} = \frac{1}{2} \left( 8 \right) = \frac{1}{2} \cdot 8 = 4 \]

For the point we already have the x-coordinate. We just need the y-coordinate.

\[ y = \sqrt{x} \bigg|_{x=64} = \sqrt{64} = 4 \]

Thus, the point is (64,4).

Finally, we use the point-slope formula.

\[ y - 4 = \frac{4}{64} (x - 64) \]

\[ y - \frac{12}{3} = \frac{1}{64} x - \frac{16}{3} \]

\[ y = \frac{1}{64} x + \frac{4}{3} \]

Therefore, the equation of the tangent line is \( y = \frac{1}{64} x + \frac{4}{3} \).

\[ \square \]

One of the problems students face in this class is keeping track of what \( y \) tells you and what \( y' \) tell you.

For \( y = f(x) \)

\[ y \text{ is the 2nd coordinate of a point on the graph, } y = f(x) \]

\[ y' \text{ is the slope of the tangent line to } y = f(x) \]

We are still very limited as to what functions we can take the derivative of. The next group will open up sums and differences of power terms and what to do with constant factors.
Basic Rules of Derivatives

For any real constant $c$ and functions $f(x)$ and $g(x)$,

\[
\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)
\]

\[
\frac{d}{dx}[c] = 0
\]

\[
\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)
\]

\[
\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)
\]

Proofs:

\[
\frac{d}{dx}[c \cdot f(x)] = \lim_{\Delta x \to 0} \frac{c \cdot f(x + \Delta x) - c \cdot f(x)}{\Delta x} = c \cdot \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = c \cdot f'(x)
\]

\[
\frac{d}{dx}[c] = \frac{d}{dx}[c \cdot 1] = c \cdot \frac{d}{dx}[1] = c \cdot 0 = 0
\]

\[
\frac{d}{dx}[f(x) + g(x)] = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = f'(x) + g'(x)
\]

\[
\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x) + (-1 \cdot g(x))] = f'(x) + \frac{d}{dx}[-1 \cdot g(x)] = f'(x) - g'(x)
\]

Example #4: Differentiate the following functions.

a) $y = 7x^3$  

\[
y' = 21x^2
\]

b) $y = x^8 - x^6$  

\[
y' = 8x^7 - 6x^5
\]

c) $y = \frac{3}{x^2} + 9x^3$  

\[
y' = -6x^{-3} + 27x^2
\]

d) $y = (x^2 - 3)(x + 1) = x^3 + x^2 - 3x - 3$  

\[
y' = 3x^2 + 2x - 3x^0 - 0 = 3x^2 + 2x - 3
\]

e) $y = \frac{x^4 + 5x^2 - 8x}{x^2}$  

\[
y' = 2x + 16x^{-1} = 2x + \frac{16}{x}
\]

Note that we had to expand part d) and break up part e) before we could apply these rules. Unlike addition and subtraction, you can’t just take the derivative of the individual parts with multiplication and division. So the following would be wrong.

\[
d) \quad y = (x^2 - 3)(x + 1) \quad \frac{dy}{dx} = (2x)(1) = 2x
\]

\[
e) \quad y = \frac{x^4 + 5x^2 - 8x}{x^2} \quad y' = 4x^3 + 10x^{-1} - \frac{8}{2x}
\]

We’ll learn how to take derivative of products and quotients later.
The Derivatives of the Cosine and Sine Functions:

<table>
<thead>
<tr>
<th>Sine and Cosine Derivatives</th>
</tr>
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<tbody>
<tr>
<td>( \frac{d}{dx}[\sin(x)] = \cos(x) )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\cos(x)] = -\sin(x) )</td>
</tr>
</tbody>
</table>

where \( x \) is in radians.

Proofs:

\[
\frac{d}{dx}[\sin(x)] = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(x)\sin(\Delta x) + \sin(x)[\cos(\Delta x) - 1]}{\Delta x} = \cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} + \sin(x) \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = \cos(x)
\]

\[
\frac{d}{dx}[\cos(x)] = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(x)\cos(\Delta x) - \sin(x)\sin(\Delta x) - \cos(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\sin(x)\sin(\Delta x) + \cos(x)[\cos(\Delta x) - 1]}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\sin(x)\sin(\Delta x)}{\Delta x} + \cos(x) \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = -\sin(x)
\]

Therefore, the shortcuts for taking the derivative of sine and cosine is nothing like for power functions, but the basic rules for constant factors, sums and differences would still apply.

Example #5: Differentiate the following functions.

a) \( y = 7\sin(x) \)

b) \( y = 9\cos(x) - x^3 \)

c) \( y = \sin(x) - 3\cos(x) \)

d) \( y = \cos\left(x - \frac{\pi}{2}\right) \)

\[
\begin{align*}
&\text{a) } y' = 7\cos(x) \\
&\text{b) } \frac{dy}{dx} = -9\sin(x) - 3x^2 \\
&\text{c) } \frac{dy}{dx} = \cos(x) + 3\sin(x) \\
&\text{d) } y = \cos\left(x - \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right) + \sin(x)\sin\left(\frac{\pi}{2}\right) = \frac{1}{2}\cos(x) + \frac{\sqrt{2}}{2}\sin(x) \\
&\quad y' = -\frac{1}{2}\sin(x) + \frac{\sqrt{2}}{2}\cos(x)
\end{align*}
\]

Example #6: If the position function of the end of a bouncing spring is given by \( P(t) = 9\cos(\omega \cdot t) \), where \( \omega = \frac{1}{\text{sec}} \), \( t \) is in seconds and \( P(t) \) is given in cm. Find the velocity of the end of the spring when \( t = 8\text{sec.} \) Round the answer to the thousandths.

First let’s note that all \( \omega = \frac{1}{\text{sec}} \) is doing is a unit conversion from seconds to radians. So if we now ignore the units and substitute in for \( \omega \) to get basically \( P(t) = 9\cos(t) \) and just know that the number inside will actually be in radians.

velocity = \( v(t) = \frac{d}{dt}[P(x)] = \frac{d}{dt}[9\cos(t)] = -9\sin(t) \)

velocity \( v(8) = -9\sin(8) \approx -8.904 \text{ cm/sec} \).

Notice that I had to make \( \omega = \frac{1}{\text{sec}} \) in the previous problem because we still don’t know how to take derivatives of a composition of 2 functions, a function within a function like \( f(x) = \cos(5x) \). We will learn this later.